

Lectures on Electromagnetic Field Theory

WENG Cho CHEW

Spring 2021,¹ Purdue University

¹Updated May 18, 2021

Contents

Preface	xii
Acknowledgements	xiii
1 Introduction, Maxwell's Equations	1
1.1 Importance of Electromagnetics	1
1.1.1 A Brief History of Electromagnetics	4
1.2 Maxwell's Equations in Integral Form	6
1.3 Static Electromagnetics	7
1.3.1 Coulomb's Law (Statics)	7
1.3.2 Electric Field (Statics)	8
1.3.3 Gauss's Law for Electric Flux (Statics)	10
1.3.4 Derivation of Gauss's Law from Coulomb's Law (Statics)	11
2 Maxwell's Equations, Differential Operator Form	15
2.1 Gauss's Divergence Theorem	15
2.1.1 Some Details	17
2.1.2 Gauss's Law in Differential Operator Form	19
2.1.3 Physical Meaning of Divergence Operator	19
2.2 Stokes's Theorem	20
2.2.1 Faraday's Law in Differential Operator Form	23
2.2.2 Physical Meaning of Curl Operator	24
2.3 Maxwell's Equations in Differential Operator Form	24
2.4 Historical Notes	25
3 Constitutive Relations, Wave Equation, and Static Green's Function	27
3.1 Simple Constitutive Relations	27
3.2 Emergence of Wave Phenomenon, Triumph of Maxwell's Equations	28
3.3 Static Electromagnetics-Revisted	32
3.3.1 Electrostatics	32
3.3.2 Poisson's Equation	33
3.3.3 Static Green's Function	34
3.3.4 Laplace's Equation	35

4 Magnetostatics, Boundary Conditions, and Jump Conditions	39
4.1 Magnetostatics	39
4.1.1 More on Coulomb Gauge	41
4.2 Boundary Conditions—1D Poisson’s Equation	41
4.3 Boundary Conditions—Maxwell’s Equations	43
4.3.1 Faraday’s Law	44
4.3.2 Gauss’s Law for Electric Flux	45
4.3.3 Ampere’s Law	46
4.3.4 Gauss’s Law for Magnetic Flux	48
5 Biot-Savart law, Conductive Media Interface, Instantaneous Poynting’s Theorem	49
5.1 Derivation of Biot-Savart Law	50
5.2 Shielding by Conductive Media	52
5.2.1 Boundary Conditions—Conductive Media Case	52
5.2.2 Electric Field Inside a Conductor	53
5.2.3 Magnetic Field Inside a Conductor	54
5.3 Instantaneous Poynting’s Theorem	56
6 Time-Harmonic Fields, Complex Power	61
6.1 Time-Harmonic Fields—Linear Systems	62
6.2 Fourier Transform Technique	64
6.3 Complex Power	65
7 More on Constitutive Relations, Uniform Plane Wave	69
7.1 More on Constitutive Relations	69
7.1.1 Isotropic Frequency Dispersive Media	69
7.1.2 Anisotropic Media	71
7.1.3 Bi-anisotropic Media	72
7.1.4 Inhomogeneous Media	72
7.1.5 Uniaxial and Biaxial Media	73
7.1.6 Nonlinear Media	73
7.2 Wave Phenomenon in the Frequency Domain	74
7.3 Uniform Plane Waves in 3D	75
8 Lossy Media, Lorentz Force Law, Drude-Lorentz-Sommerfeld Model	79
8.1 Plane Waves in Lossy Conductive Media	79
8.1.1 Highly Conductive Case	81
8.1.2 Lowly Conductive Case	81
8.2 Lorentz Force Law	82
8.3 Drude-Lorentz-Sommerfeld Model	82
8.3.1 Cold Collisionless Plasma Medium	83
8.3.2 Bound Electron Case	85
8.3.3 Damping or Dissipation Case	86
8.3.4 Broad Applicability of Drude-Lorentz-Sommerfeld Model	87

CONTENTS	iii
8.3.5 Frequency Dispersive Media	89
8.3.6 Plasmonic Nanoparticles	90
9 Waves in Gyrotropic Media, Polarization	93
9.1 Gyrotropic Media and Faraday Rotation	93
9.2 Wave Polarization	96
9.2.1 General Polarizations—Elliptical and Circular Polarizations	96
9.2.2 Arbitrary Polarization Case and Axial Ratio ¹	98
9.3 Polarization and Power Flow	100
10 Momentum, Complex Poynting's Theorem, Lossless Condition, Energy Density	103
10.1 Spin Angular Momentum and Cylindrical Vector Beam	104
10.2 Momentum Density of Electromagnetic Field	105
10.3 Complex Poynting's Theorem and Lossless Conditions	106
10.3.1 Complex Poynting's Theorem	106
10.3.2 Lossless Conditions	107
10.4 Energy Density in Dispersive Media ²	109
11 Transmission Lines	113
11.1 Transmission Line Theory	114
11.1.1 Time-Domain Analysis	115
11.1.2 Frequency-Domain Analysis—the Power of Phasor Technique Again! .	118
11.2 Lossy Transmission Line	120
12 More on Transmission Lines	123
12.1 Terminated Transmission Lines	123
12.1.1 Shorted Terminations	126
12.1.2 Open Terminations	127
12.2 Smith Chart	128
12.3 VSWR (Voltage Standing Wave Ratio)	130
13 Multi-Junction Transmission Lines, Duality Principle	137
13.1 Multi-Junction Transmission Lines	137
13.1.1 Single-Junction Transmission Lines	139
13.1.2 Two-Junction Transmission Lines	140
13.1.3 Stray Capacitance and Inductance	143
13.1.4 Multi-Port Network	144
13.2 Duality Principle	145
13.2.1 Unusual Swaps ³	146
13.2.2 Left-Handed Materials and Double Negative Materials	146
13.3 Fictitious Magnetic Currents	147

¹This section is mathematically complicated. It can be skipped on first reading.

²The derivation in this section is complex, but worth the pain, since this knowledge was not discovered until the 1960s.

³This section can be skipped on first reading.

14 Reflection, Transmission, and Interesting Physical Phenomena	149
14.1 Reflection and Transmission—Single Interface Case	149
14.1.1 TE Polarization (Perpendicular or E Polarization) ⁴	150
14.1.2 TM Polarization (Parallel or H Polarization) ⁵	153
14.1.3 Lens Optics and Ray Tracing	153
14.2 Interesting Physical Phenomena	154
14.2.1 Total Internal Reflection	155
15 Brewster Angle, SPP, Homomorphism with Transmission Lines	161
15.1 Brewster's Angle	161
15.1.1 Surface Plasmon Polariton	164
15.2 Homomorphism of Uniform Plane Waves and Transmission Lines Equations .	166
15.2.1 TE or TE_z Waves	167
15.2.2 TM or TM_z Waves	168
16 Waves in Layered Media	171
16.1 Waves in Layered Media	171
16.1.1 Generalized Reflection Coefficient for Layered Media	172
16.1.2 Ray Series Interpretation of Generalized Reflection Coefficient	173
16.2 Phase Velocity and Group Velocity	174
16.2.1 Phase Velocity	174
16.2.2 Group Velocity	175
16.3 Wave Guidance in a Layered Media	179
16.3.1 Transverse Resonance Condition	179
17 Dielectric Slab Waveguides	181
17.1 Generalized Transverse Resonance Condition	181
17.1.1 Parallel Plate Waveguide	182
17.2 Dielectric Slab Waveguide	182
17.2.1 TE Case	183
17.2.2 TM Case	189
17.2.3 A Note on Cut-Off of Dielectric Waveguides	190
18 Hollow Waveguides	191
18.1 General Information on Hollow Waveguides	191
18.1.1 Absence of TEM Mode in a Hollow Waveguide	192
18.1.2 TE Case ($E_z = 0, H_z \neq 0$, TE_z case)	193
18.1.3 TM Case ($E_z \neq 0, H_z = 0$, TM_z Case)	195
18.2 Rectangular Waveguides	196
18.2.1 TE Modes ($H_z \neq 0$, H Modes or TE_z Modes)	196

⁴These polarizations are also variously known as TE_z , or the *s* and *p* polarizations, a descendent from the notations for acoustic waves where *s* and *p* stand for shear and pressure waves, respectively.

⁵Also known as TM_z polarization.

19 More on Hollow Waveguides	199
19.1 Rectangular Waveguides, Contd.	200
19.1.1 TM Modes ($E_z \neq 0$, E Modes or TM_z Modes)	200
19.1.2 Bouncing Wave Picture	201
19.1.3 Field Plots	202
19.2 Circular Waveguides	204
19.2.1 TE Case	204
19.2.2 TM Case	206
20 More on Waveguides and Transmission Lines	211
20.1 Circular Waveguides, Contd.	211
20.1.1 An Application of Circular Waveguide	212
20.2 Remarks on Quasi-TEM Modes, Hybrid Modes, and Surface Plasmonic Modes	215
20.2.1 Quasi-TEM Modes	216
20.2.2 Hybrid Modes—Inhomogeneously-Filled Waveguides	217
20.2.3 Guidance of Modes	218
20.3 Homomorphism of Waveguides and Transmission Lines	219
20.3.1 TE Case	219
20.3.2 TM Case	221
20.3.3 Mode Conversion	222
21 Cavity Resonators	225
21.1 Transmission Line Model of a Resonator	225
21.2 Cylindrical Waveguide Resonators	228
21.2.1 $\beta_z = 0$ Case for Cylindrical Waveguides	230
21.2.2 Lowest Mode of a Rectangular Cavity	230
21.3 Some Applications of Resonators	232
21.3.1 Filters	232
21.3.2 Electromagnetic Sources	234
21.3.3 Frequency Sensor	237
22 Quality Factor of Cavities, Mode Orthogonality	239
22.1 The Quality Factor of a Cavity—General Concept	239
22.1.1 Analogue with an LC Tank Circuit	240
22.1.2 Relation to Bandwidth and Pole Location	243
22.1.3 Wall Loss and \mathbf{Q} for a Metallic Cavity—A Perturbation Concept	244
22.1.4 Example: The Q of TM_{110} Mode	246
22.2 Mode Orthogonality and Matrix Eigenvalue Problem	248
22.2.1 Matrix Eigenvalue Problem (EVP)	248
22.2.2 Homomorphism with the Waveguide Mode Problem	249
22.2.3 Proof of Orthogonality of Waveguide Modes ⁶	250

⁶This may be skipped on first reading.

23 Scalar and Vector Potentials	253
23.1 Scalar and Vector Potentials for Time-Harmonic Fields	253
23.2 Scalar and Vector Potentials for Statics—A Review	254
23.2.1 Scalar and Vector Potentials for Electrodynamics	255
23.2.2 More on Scalar and Vector Potentials	257
23.3 When is Static Electromagnetic Theory Valid?	258
23.3.1 Cutting Through The Chaste	259
23.3.2 Dimensional Analysis Approach and Coordinate Stretching ⁷	260
23.3.3 Quasi-Static Electromagnetic Theory	264
24 Circuit Theory Revisited	267
24.1 Kirchhoff Current Law	267
24.2 Kirchhoff Voltage Law	268
24.3 Inductor	272
24.4 Capacitance	273
24.5 Resistor	274
24.6 Some Remarks	274
24.7 Energy Storage Method for Inductor and Capacitor	275
24.8 Finding Closed-Form Formulas for Inductance and Capacitance	275
24.9 Importance of Circuit Theory in IC Design	278
24.10 Decoupling Capacitors and Spiral Inductors	280
25 Radiation by a Hertzian Dipole	283
25.1 History	283
25.2 Approximation by a Point Source	285
25.2.1 Case I. Near Field, $\beta r \ll 1$	287
25.2.2 Case II. Far Field (Radiation Field), $\beta r \gg 1$	288
25.3 Radiation, Power, and Directive Gain Patterns	289
25.3.1 Radiation Resistance	291
26 Radiation Fields, Directive Gain, Effective Aperture	295
26.1 Radiation Fields or Far-Field Approximation	297
26.1.1 Far-Field Approximation	298
26.1.2 Locally Plane Wave Approximation	299
26.1.3 Directive Gain Pattern Revisited	302
26.2 Effective Aperture and Directive Gain	303
27 Array Antennas, Fresnel Zone, Rayleigh Distance	305
27.1 Linear Array of Dipole Antennas	305
27.1.1 Far-Field Approximation of a Linear Array	307
27.1.2 Radiation Pattern of an Array	307
27.2 Validity of the Far-Field Approximation	310
27.2.1 Rayleigh Distance	312
27.2.2 Near Zone, Fresnel Zone, and Far Zone	313

⁷This can be skipped on first reading.

28 Different Types of Antennas—Heuristics	315
28.1 Resonance Tunneling in Antenna	316
28.2 Horn Antennas	319
28.3 Quasi-Optical Antennas	321
28.4 Small Antennas	323
29 Uniqueness Theorem	329
29.1 The Difference Solutions to Source-Free Maxwell's Equations	329
29.2 Conditions for Uniqueness	332
29.2.1 Isotropic Case	332
29.2.2 General Anisotropic Case	333
29.3 Hind Sight Using Linear Algebra	334
29.4 Connection to Poles of a Linear System	335
29.5 Radiation from Antenna Sources and Radiation Condition	337
30 Reciprocity Theorem	341
30.1 Mathematical Derivation	342
30.1.1 Lorentz Reciprocity Theorem	344
30.1.2 Reaction Reciprocity Theorem	344
30.2 Conditions for Reciprocity	345
30.3 Application to a Two-Port Network and Circuit Theory	345
30.4 Voltage Sources in Electromagnetics	348
30.5 Hind Sight	349
30.6 Transmit and Receive Patterns of an Antennna	350
30.6.1 Effective Gain versus Directive Gain	350
30.6.2 Effective Aperture	351
31 Equivalence Theorems, Huygens' Principle	355
31.1 Equivalence Theorems or Equivalence Principles	355
31.1.1 Inside-Out Case	356
31.1.2 Outside-in Case	357
31.1.3 General Case	357
31.2 Electric Current on a PEC	358
31.3 Magnetic Current on a PMC	359
31.4 Huygens' Principle and Green's Theorem	359
31.4.1 Scalar Waves Case	360
31.4.2 Electromagnetic Waves Case	363
32 Shielding, Image Theory	367
32.1 Shielding	367
32.1.1 A Note on Electrostatic Shielding	367
32.1.2 Relaxation Time	368
32.2 Image Theory	370
32.2.1 Electric Charges and Electric Dipoles	370
32.2.2 Magnetic Charges and Magnetic Dipoles	372

32.2.3	Perfect Magnetic Conductor (PMC) Surfaces	374
32.2.4	Multiple Images	375
32.2.5	Some Special Cases—Spheres, Cylinders, and Dielectric Interfaces	376
33	High Frequency Solutions, Gaussian Beams	379
33.1	Tangent Plane Approximations	380
33.2	Fermat’s Principle	381
33.2.1	Generalized Snell’s Law	383
33.3	Gaussian Beam	384
33.3.1	Derivation of the Paraxial/Parabolic Wave Equation	384
33.3.2	Finding a Closed Form Solution	385
33.3.3	Other solutions	387
34	Scattering of Electromagnetic Field	391
34.1	Rayleigh Scattering	391
34.1.1	Scattering by a Small Spherical Particle	393
34.1.2	Scattering Cross Section	395
34.1.3	Small Conductive Particle	398
34.2	Mie Scattering	399
34.2.1	Optical Theorem	400
34.2.2	Mie Scattering by Spherical Harmonic Expansions	401
34.2.3	Separation of Variables in Spherical Coordinates ⁸	401
35	Spectral Expansions of Source Fields—Sommerfeld Integrals	405
35.1	Spectral Representations of Sources	405
35.1.1	A Point Source—Fourier Expansion and Contour Integration	406
35.2	A Source on Top of a Layered Medium	411
35.2.1	Electric Dipole Fields—Spectral Expansion	412
35.3	Stationary Phase Method—Fermat’s Principle	414
36	Computational Electromagnetics, Numerical Methods	419
36.1	Computational Electromagnetics, Numerical Methods	421
36.2	Examples of Differential Equations	421
36.3	Examples of Integral Equations	422
36.3.1	Volume Integral Equation	422
36.3.2	Surface Integral Equation	424
36.4	Function as a Vector	425
36.5	Operator as a Map	426
36.5.1	Domain and Range Spaces	426
36.6	Approximating Operator Equations with Matrix Equations	427
36.6.1	Subspace Projection Methods	427
36.6.2	Dual Spaces	428
36.6.3	Matrix and Vector Representations	428
36.6.4	Mesh Generation	429

⁸May be skipped on first reading.

36.6.5	Differential Equation Solvers versus Integral Equation Solvers	430
36.7	Solving Matrix Equation by Optimization	431
36.7.1	Gradient of a Functional	432
37	Finite Difference Method, Yee Algorithm	435
37.1	Finite-Difference Time-Domain Method	435
37.1.1	The Finite-Difference Approximation	436
37.1.2	Time Stepping or Time Marching	438
37.1.3	Stability Analysis	440
37.1.4	Grid-Dispersion Error	442
37.2	The Yee Algorithm	444
37.2.1	Finite-Difference Frequency Domain Method	448
37.3	Absorbing Boundary Conditions	448
38	Quantum Theory of Light	451
38.1	Historical Background on Quantum Theory	451
38.2	Connecting Electromagnetic Oscillation to Simple Pendulum	454
38.3	Hamiltonian Mechanics	457
38.4	Schrödinger Equation (1925)	459
38.5	Some Quantum Interpretations—A Preview	462
38.5.1	Matrix or Operator Representations	463
38.6	Bizarre Nature of the Photon Number States	464
39	Quantum Coherent State of Light	467
39.1	The Quantum Coherent State	467
39.1.1	Quantum Harmonic Oscillator Revisited—Creation and Annihilation Operators	468
39.2	Some Words on Quantum Randomness and Quantum Observables	470
39.3	Derivation of the Coherent States	471
39.3.1	Time Evolution of a Quantum State	473
39.4	More on the Creation and Annihilation Operator	475
39.4.1	The Correspondence Principle for a Pendulum	476
39.4.2	Connecting Quantum Pendulum to Electromagnetic Oscillator ⁹	478
39.5	Epilogue	480

⁹May be skipped on first reading.

Preface

This set of lecture notes is from my teaching of ECE 604, Electromagnetic Field Theory, at ECE, Purdue University, West Lafayette. It is intended for entry level graduate students. Because different universities have different undergraduate requirements in electromagnetic field theory, this is a course intended to “level the playing field”. From this point onward, hopefully, all students will have the fundamental background in electromagnetic field theory needed to take advance level courses and do research at Purdue.

In developing this course, I have drawn heavily upon knowledge of our predecessors in this area. Many of the textbooks and papers used, I have listed them in the reference list. Being a practitioner in this field for over 40 years, I have seen electromagnetic theory impacting modern technology development unabated. Despite its age, the set of Maxwell’s equations has endured and continued to be important, from statics to optics, from classical to quantum, and from nanometer lengthscales to galactic lengthscales. The applications of electromagnetic technologies have also been tremendous and wide-ranging: from geophysical exploration, remote sensing, bio-sensing, electrical machinery, renewable and clean energy, biomedical engineering, optics and photonics, computer chip, computer system, and quantum computer designs, quantum communication and many more. Electromagnetic field theory is not everything, but it remains an important component of modern technology developments.

The challenge in teaching this course is on how to teach over 150 years of knowledge in one semester: Of course this is mission impossible! To do this, we use the traditional wisdom of engineering education: Distill the knowledge, make it as simple as possible, and teach the fundamental big ideas in one short semester. Because of this, you may find the flow of the lectures erratic. Some times, I feel the need to touch on certain big ideas before moving on, resulting in the choppiness of the curriculum.

Also, in this course, I exploit mathematical homomorphism as much as possible to simplify the teaching. After years of practising in this area, I find that some complex and advanced concepts become simpler if mathematical homomorphism is exploited between the advanced concepts and simpler ones. An example of this is on waves in layered media. The problem is homomorphic to the transmission line problem: Hence, using transmission line theory, one can simplify the derivations of some complicated formulas.

A large part of modern electromagnetic technologies is based on heuristics. This is something difficult to teach, as it relies on physical insight and experience. Modern commercial software has reshaped this landscape, as the field of math-physics modeling through numerical simulations, known as computational electromagnetic (CEM), has made rapid advances in recent years. Many cut-and-try laboratory experiments, based on heuristics, have been

replaced by cut-and-try computer experiments, which are a lot cheaper.

An exciting modern development is the role of electromagnetics and Maxwell's equations in quantum technologies. We will connect Maxwell's equations to them toward the end of this course. This is a challenge, as it has never been done before at an entry level course to my knowledge.

*Weng Cho CHEW
May 18, 2021 Purdue University*

Acknowledgements

I like to thank Dan Jiao for sharing her lecture notes in this course, as well as Andy Weiner for sharing his experience in teaching this course. Mark Lundstrom gave me useful feedback on Lectures 38 and 39 on the quantum theory of light. Also, I am thankful to Dong-Yeop Na for helping teach part of this course. Robert Hsueh-Yung Chao also took time to read the lecture notes and gave me very useful feedback.